# 't Hooft Loops, Electric Flux Sectors and Confinement in $SU(2)$ Yang-Mills Theory

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We use 't Hooft loops of maximal size on finite lattices to calculate the free energy in the sectors of $SU(2)$ Yang-Mills theory with fixed electric flux as a function of temperature and (spatial) volume. Our results provide evidence for the mass gap. The confinement of electric fluxes in the low temperature phase and their condensation in the high temperature phase are demonstrated. In a surprisingly large scaling window around criticality, the transition is quantitatively well described by universal exponents and amplitude ratios relating the properties of the two phases.

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Center symmetry is widely believed to play a key role for confinement, which has been emphasized by 't Hooft's introduction of the dual to the Wilson loop [1]. Temporal 't Hooft loops in $SU(2)$ show screening for the interaction of a static pair of center monopoles in both phases, for $T < T_c$ and for $T > T_c$ [2,3], just as spatial Wilson loops exhibit an area law in either case. The expectation values of (sufficiently large) spatial 't Hooft loops $\widetilde{W}(C)$, on the other hand, change from a screening behavior below $T_c$ to a confined one [4] with a dual string tension $\tilde{\sigma}$ and an area law in the electrically deconfined phase above $T_c$ [3],

$$\langle \widetilde{W}(C) \rangle \sim \exp\{-\tilde{\sigma}(T)\, LR\} , \quad \text{at } T > T_c , \tag{1}$$

for a rectangular curve $C$ spanning a spatial surface of size $L \times R$ on a hypercubic $1/T \times L^3$ lattice.

Ref. [3] also confirmed numerically a perimeter law for 't Hooft loops at $T = 0$. The qualitative behavior of the spatial 't Hooft loops is thus the same as that of the Wilson loops in the 3-dimensional $\mathbb{Z}_2$ gauge theory, the dual to the 3-dimensional Ising model [5]. Furthermore, as the phase transition is approached (from above), the temperature dependence of the $SU(2)$ dual string tension resembles that of the interface tension in the 3-dimensional Ising model (below $T_c$), as expected from universality.

To introduce a spatial 't Hooft loop of maximal size $L \times L$, living in, say, the $(x, y)$ plane of the dual lattice, one multiplies one plaquette in every $(z, t)$ plane of the original lattice by a non-trivial element of the center of $SU(N)$ in such a way that the modified plaquettes form a coclosed set. This creates a $\mathbb{Z}_N$-interface which is equivalent to enforcing boundary conditions with twist in the $(z, t)$ directions. Combining two and three maximal $L \times L$ 't Hooft loops in orthogonal spatial planes yields the partition functions of $SU(N)$ Yang-Mills theory on finite lattices for all possible combinations of temporal twists. From these, one obtains the free energies in the presence of fixed units of electric flux $\vec{e} \in \mathbb{Z}_N^3$ via a $\mathbb{Z}_N$ Fourier transform as shown by 't Hooft [6]. This leads to the respective electric-flux superselection sectors (with vanishing magnetic flux and vacuum angle) of the theory in the thermodynamic limit. In this letter, we verify for $SU(2)$ that the partition functions of electric fluxes $\vec{e} \neq 0$ vanish exponentially with the spatial size $L$ of the system in the magnetic Higgs phase with electric confinement for $T < T_c$, whereas for $T > T_c$ they become equal to that of the neutral $\vec{e} = 0$ sector. Describing the transition by critical exponents of the 3d-Ising class, we discuss how universal amplitude ratios relate the two phases.

*Twisted Boundary Conditions and Electric Fluxes*

On the 4-dimensional torus $T^4$ there are $N^6$ gauge non-equivalent choices for imposing periodic boundary conditions on the $SU(N)$ gauge potentials which are invariant under the center $\mathbb{Z}_N$ of $SU(N)$. In particular, the general boundary conditions for the potentials $A$ to be periodic in each direction $\mu$ up to twists $\Omega_\mu \in SU(N)$,

$$A(x + L_\mu) = \Omega_\mu(x) A(x) \Omega_\mu^{-1}(x) + \Omega_\mu(x) d\Omega_\mu^{-1}(x) , \tag{2}$$

where $L_\mu$ has the length of the torus in that direction, specify the connections in the $SU(N)/\mathbb{Z}_N$ bundles over the torus. The cocycle condition for the twists ensures compatibility of successive translations in $(\mu, \nu)$ planes:

$$\Omega_\mu(x + L_\nu)\Omega_\nu(x) = Z_{\mu\nu}\, \Omega_\nu(x + L_\mu)\Omega_\mu(x) \;\; \text{(no sum)}$$
$$\text{with } \; Z_{\mu\nu} = e^{2\pi i n_{\mu\nu}/N} \; , \text{ and } n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}_N \, . \tag{3}$$

Selecting a time/temperature direction, the $n_{0i} \equiv k_i$ then determine *temporal twists*, while the *spatial twists* $n_{ij} \equiv \epsilon_{ijk} m_k$ are directly given in terms of the conserved, $\mathbb{Z}_N$-valued magnetic flux $m_k$ through the box in the $k$-direction. To fully classify the bundles over $T^4$ one shows [7] that the Pontryagin index,

$$P = \frac{1}{16\pi^2} \int_{T^4} \operatorname{tr} F\widetilde{F} \, = \nu + \frac{\vec{k} \cdot \vec{m}}{N} \, , \tag{4}$$

is determined by these six $\mathbb{Z}_N$ twists $\vec{k}$, $\vec{m}$ and the instanton winding number $\nu \in \mathbb{Z}$.

To obtain the electric flux sectors, one performs a $\mathbb{Z}_N$ Fourier transform w.r.t. the temporal twist. For non-zero vacuum angle $\theta$ the general expression for the free energy $F(\vec{e},\vec{m},\theta)$ in a sector of given electric and magnetic fluxes $\vec{e}$ and $\vec{m}$, respectively, (and fixed $\theta$) reads [6],

$$e^{-\frac{1}{T}F(\vec{e},\vec{m},\theta)} = \frac{1}{N^3}\sum_{\{k_i=0\}}^{N-1}\sum_{\nu=-\infty}^{\infty} e^{-2\pi i\vec{e}\cdot\vec{k}/N} \times e^{i\theta(\nu+\vec{k}\cdot\vec{m}/N)}\, Z(\vec{k},\vec{m},\nu)\,, \qquad (5)$$

in which $Z(\vec{k},\vec{m},\nu)$ is the (finite volume) partition function for fixed twists and winding number.

To calculate $Z(\vec{k},\vec{m}) \equiv \sum_\nu Z(\vec{k},\vec{m},\nu)$ (for $\theta = 0$) on a $1/T \times L^3$ lattice for $SU(2)$ (with $k_i$ and $m_i$ in $\{0,1\}$) we introduce couplings with reversed signs into the usual Wilson action by replacing $\beta \to -\beta$ for one plaquette in every $(\mu,\nu)$ plane corresponding to an $n_{\mu\nu}$-twist,

$$S(\beta,\vec{k},\vec{m}) = -\sum_P \beta(P)\,\frac{1}{2}\,\mathrm{Tr}(U_P)\ . \qquad (6)$$

Herein, the sum extends over all plaquettes $P$ with $U_P$ denoting the path-ordered product of the links around $P$ and we introduced a plaquette-dependent coupling,

$$\beta(P) = \begin{cases} -\beta\,, & P \in \mathcal{P}(n_{\mu\nu}) \\ \ \ \beta\,, & P \notin \mathcal{P}(n_{\mu\nu}) \end{cases} \qquad (7)$$

where $\mathcal{P}(n_{\mu\nu})$ denotes the coclosed stacks of plaquettes dual to the planes of the maximal 't Hooft loops. These stacks of flipped plaquettes create $\mathbb{Z}_2$ interfaces and are equivalent to enforcing twist in the $(\mu,\nu)$ directions orthogonal to the planes of the respective 't Hooft loops.

The partition functions of the twist sectors, relative to the untwisted $Z_\beta$ (such that $Z_\beta(\vec{0},\vec{0}) = 1$), are then:

$$Z_\beta(\vec{k},\vec{m}) = Z_\beta^{-1}\int[dU]\,\exp -S(\beta,\vec{k},\vec{m})\ . \qquad (8)$$

Accordingly, flipping the couplings of plaquettes dual to a given surface subtended by some closed curve $C$ yields the expectation value of a finite 't Hooft loop $\widetilde{W}(C)$ in the $SU(2)$ Yang-Mills theory. An entirely analogous procedure can be used for Wilson loops in the 3-dimensional $\mathbb{Z}_2$ gauge theory. Through duality, their expectation values can be expressed as ratios of Ising-model partition functions with and without antiferromagnetic bonds at those links of the Ising-model that are dual (in 3 dimensions) to some surface spanned by the $\mathbb{Z}_2$-Wilson loop [5]. In both cases, of course, the surface is arbitrary except for its boundary. The 't Hooft loop can thus be viewed as a gauge-invariant operator which creates a fluctuating center-vortex surface with pinned boundary.

The expectation values of maximal-size 't Hooft loops, given by the partition functions of the twist sectors $Z_\beta(\vec{k},\vec{m})$, can now be used to calculate the free energies of electric and magnetic fluxes as per Eq. (5). For purely temporal twists in particular, the free energies of the electric fluxes through the $L^3$ box at temperature $T$, $F_e(\vec{e};L,T) \equiv F(\vec{e},\vec{m}=0,\theta=0) - F(\vec{e}=0,\vec{m}=0,\theta=0)$ and with $Z_k(\vec{k}) \equiv Z_\beta(\vec{k},\vec{m}=0)$ in (8), are given by:

$$Z_e(\vec{e}) = e^{-\frac{1}{T}F_e(\vec{e};L,T)} = \frac{\sum_{\{k_i=0\}}^{N-1} e^{-2\pi i\,\vec{e}\cdot\vec{k}/N}\,Z_k(\vec{k})}{\sum_{\{k_i=0\}}^{N-1} Z_k(\vec{k})} \qquad (9)$$

with $Z_e(\vec{0}) = Z_k(\vec{0}) = 1$. Because of the invariance under spatial $\pi/2$ rotations, we can write for $SU(2)$: $Z_k(1)$ if $\vec{k} = \{(1,0,0),(0,1,0),(0,0,1)\}$; $Z_k(2)$ if $\vec{k} = \{(1,1,0),(1,0,1),(0,1,1)\}$; and $Z_k(3) = Z_k(1,1,1)$, for the partition functions with one, two and three maximal 't Hooft loops in orthogonal spatial planes, respectively. With analogous notations for the $Z_e(\vec{e})$ one thus obtains:

$$Z_e(1) = \frac{1}{\mathcal{N}_e}\Big(1 + Z_k(1) - Z_k(2) - Z_k(3)\Big)\,, \qquad (10)$$

$$Z_e(2) = \frac{1}{\mathcal{N}_e}\Big(1 - Z_k(1) - Z_k(2) + Z_k(3)\Big)\,, \qquad (11)$$

$$Z_e(3) = \frac{1}{\mathcal{N}_e}\Big(1 - 3Z_k(1) + 3Z_k(2) - Z_k(3)\Big)\,, \qquad (12)$$

$$\text{with}\qquad \mathcal{N}_e \;=\; 1 + 3Z_k(1) + 3Z_k(2) + Z_k(3)\ .$$

Eqs.(10)-(12) are readily inverted via inverse $\mathbb{Z}_2$ Fourier transform, which in effect interchanges $Z_e(i) \leftrightarrow Z_k(i)$.

We now establish the connection with Polyakov loops. First, recall that the gauge invariant definition of the latter in the presence of temporal twists is given by [8],

$$P(\vec{x}) = \frac{1}{N}\mathrm{tr}\left(\mathcal{P}e^{ig\int_0^{1/T} A_0(\vec{x},t)dt}\,\Omega_t(\vec{x})\right)\ . \qquad (13)$$

Successively transforming the path-ordered exponential by the various spatial transition functions which accompany the possibly non-trivial $\Omega_t(\vec{x})$ for the transition in the time direction [9], we derive from (2) and (3) that

$$P(\vec{x}+L\vec{e}) \;=\; e^{2\pi i\,\vec{e}\cdot\vec{k}/N}\,P(\vec{x})\ ,\ \text{ or}$$
$$P(\vec{x})P^\dagger(\vec{x}+L\vec{e}) \;=\; e^{-2\pi i\,\vec{e}\cdot\vec{k}/N}\,\mathbb{1}\ . \qquad (14)$$

Therefore, the electric-flux partition functions Eq. (9) are in fact the expectation values of Polyakov loop correlators in the ensemble average over all temporal twists with the enlarged partition function $Z = \sum_{\{k_i=0\}}^{N-1} Z_k(\vec{k})$,

$$Z_e(\vec{e}) = e^{-\frac{1}{T}F_e(\vec{e};L,T)} = \big\langle P(\vec{x})P^\dagger(\vec{x}+L\vec{e})\big\rangle_{L,T}\ . \qquad (15)$$

Eq. (9) thus yields a dual relation between Polyakov loops and temporal twists according to the general pattern,

$$\big\langle P(\vec{x})P^\dagger(\vec{x}+L\vec{e})\big\rangle \overset{L\to\infty}{\longrightarrow} \begin{cases} 0\,,\ \text{for } Z_k(\vec{k}) \to 1\,,\ T < T_c \\ 1\,,\ \text{for } Z_k(\vec{k}) \to 0\,,\ T > T_c \end{cases}$$

reflecting the different realizations of the electric $\mathbb{Z}_N$ center symmetry in the respective phases.

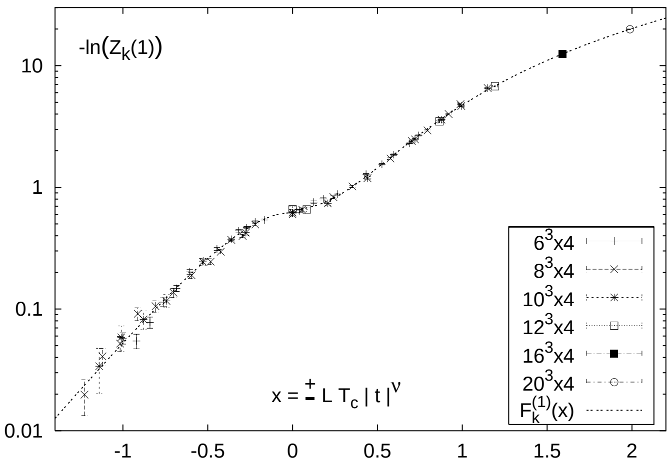


FIG. 1. The free energy of one temporal twist as a function of the finite size scaling variable $x$ (with $x<0$ for $T<T_c$).

*Finite Size Scaling and Universal Amplitude Ratios*

We compute the three partition functions $Z_k(i)$ for $SU(2)$ by Monte Carlo, using the method of Ref. [3]. For pioneering related work at $T=0$, see Ref. [10]. The $Z_k(i)$ are the analogues of ratios of 3d-Ising model partition functions with different boundary conditions. As for the latter [11], we assume their $L$, $T$ dependence to be governed by simple finite-size scaling laws,

$$Z_k(i) \;=\; f_{\pm}^{(i)}(x)\;,\; i=1,..3\;. \tag{16}$$

The $f_{\pm}^{(i)}$ are functions of the finite-size scaling variable

$$x=\pm L T_c |t|^{\nu} \propto L/\xi_{\pm}(t)\;,\;\text{for } T \gtrless T_c\;, \tag{17}$$

where $t=T/T_c-1$ and $\xi_{\pm}(t)=\xi_{\pm}^{0}|t|^{\nu}$ are the reduced temperature and the correlation lengths, respectively, and we use the exponent $\nu=0.63$ from the Ising model. The results for all different lattice sizes nicely collapse on a single curve as can be seen for one spatial 't Hooft loop in Fig. 1. We fit the free energies with an Ansatz, where the last terms model the small-size deviation from asymptotics in each phase ($a_{\pm}^{(i)}\sim c_{\pm}^{(i)}<b_{\pm}^{(i)}$; $d_{\pm}^{(i)}<1$),

$$F_k^{(i)}(x)=\begin{cases} \exp\left\{b_{-}^{(i)}x+a_{-}^{(i)}-\dfrac{c_{-}^{(i)}}{(d_{-}^{(i)}-x)^2}\right\},\; x<0 \\ b_{+}^{(i)}x^2-a_{+}^{(i)}+\dfrac{c_{+}^{(i)}}{d_{+}^{(i)}+x^2}\;,\;\; x>0 \end{cases} \tag{18}$$

and $-\ln f_{\pm}^{(i)}(x)\equiv F_k^{(i)}(x)$ for $x\gtrless 0$. Plotted over temperature, the data of Fig. 1 and the unique function $F_k^{(1)}(x)$ lead to the family of curves shown in Fig. 2 (top) in which the phase transition is exhibited most clearly. The amplitudes relevant to the large size behavior of the free energy near criticality come out as $b_{-}^{(1)}=3.87\pm 0.5$ and $b_{+}^{(1)}=\tilde{\sigma}_0^{(1)}=5.36\pm 0.1$, with additional systematic uncertainties inherent in the form of our Ansatz (18). With analogous data and fits for $Z_k(2)$ and $Z_k(3)$, from Eq. (10), we obtain $Z_e(1)$ as shown in Fig. 2 (bottom).

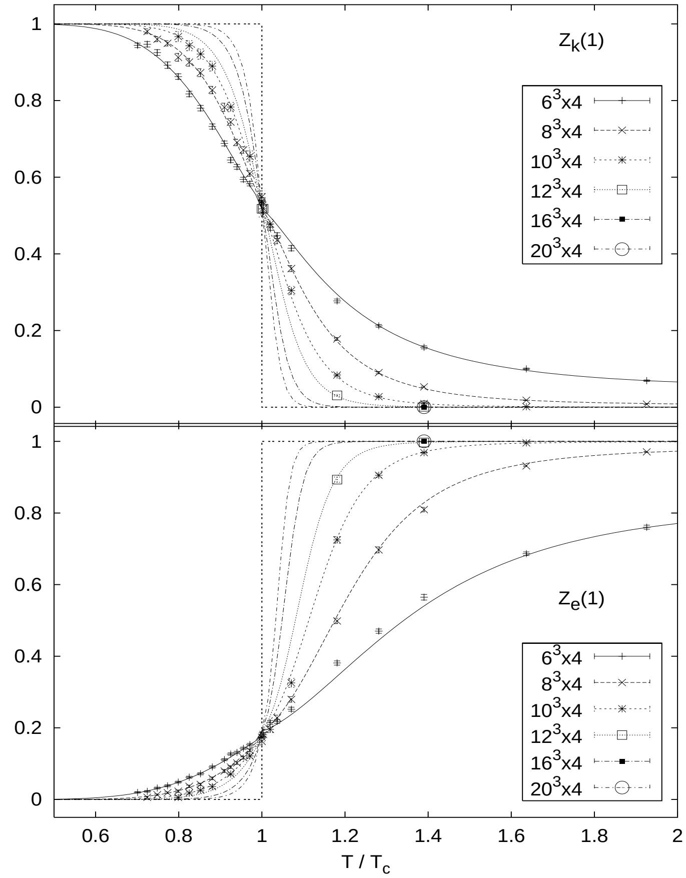


FIG. 2. The partition functions of one temporal twist (top) and one electric flux (bottom) over $T$ for the various lattices.

Corrections to scaling do not become appreciable up to $T\sim 2T_c$ indicating a surprisingly large scaling window.

Above $T_c$, the dual string tension is (with $\tilde{\sigma}_0^{(1)}=b_{+}^{(1)}$),

$$\tilde{\sigma}(T)=\tilde{\sigma}_0^{(1)}\,T_c^2\,|t|^{2\nu}=R/\xi_{+}^2(t)\;, \tag{19}$$

where the universal ratio $R\simeq 0.104$ [12,13] is known from the 3d-Ising model. There, $R=\xi_{-}^2\sigma_I$ relates the correlation length and the interface tension $\sigma_I$ for $T<T_c$. Here, Eq. (19) determines the screening length for the Polyakov loops above $T_c$, $\xi_{+}(t)=\sqrt{R/\tilde{\sigma}(T)}$. In addition, the universality hypothesis relates the ratio of the correlation lengths for the Polyakov loops in $SU(2)$ to that of their dual analogue, the correlation lengths of the spins in the 3d-Ising model, as measured in Ref. [13],

$$\xi_{-}^{SU(2)}/\xi_{+}^{SU(2)}\stackrel{!}{=}\xi_{+}^{\text{Ising}}/\xi_{-}^{\text{Ising}}\simeq 1.96\,. \tag{20}$$

Together with (19) this allows to relate the string tension amplitude below $T_c$ to its dual counter part above $T_c$. From the linear part of the electric-flux free energy,

$$\big\langle P(\vec{x})P^{\dagger}(\vec{x}+L\vec{e}_i)\big\rangle\to e^{-\sigma(T)L/T}=e^{-L/\xi_{-}(t)},\;\; T<T_c$$
$$\Rightarrow\; -\ln(Z_e(1))\to -x/(\xi_{-}^{(0)}T_c)\;,\; -x=LT_c|t|^{\nu}, \tag{21}$$

for large $L$ (or $(-x)$ large). Thus, from Eqs. (19), (20),

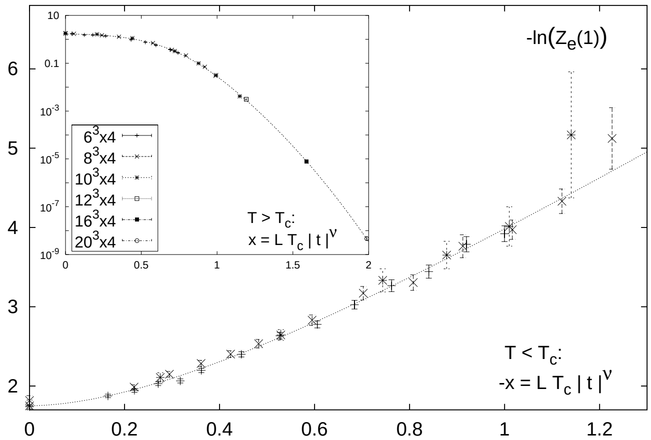


FIG. 3. Free energy of one unit of electric flux from Eq. 10 as a function of $x$ in the confined phase, and above $T_c$ (insert).

$$\frac{\sigma(T)}{T} = \frac{1}{\xi_-(t)} = T_c\,|t|^\nu \sqrt{\tilde{\sigma}_0^{(1)}/R_+}\ ,\ R_+ = \frac{\xi_-^2}{\xi_+^2}\,R \simeq 0.4$$

A value of about $\tilde{\sigma}_0^{(1)} \simeq 5.36$ then implies for the string-tension amplitude $1/(\xi_-^{(0)}T_c) \simeq 3.66$. This value was used in the fit for the slope of the linear part of $-\ln(Z_e(1))$ shown in Fig. 3. Fitting the slope to the data yields the consistent value $1/(\xi_-^{(0)}T_c) = 3.58 \pm 0.5$.

Fig. 3 shows the quality of the finite size scaling for the free energy of one electric flux, in particular, for the more accurate data from the high temperature phase (insert).

Similar results are obtained from Eqs. (11), (12) also for 2 and 3 orthogonal electric fluxes which we verify to be suppressed more strongly in the confined phase, $Z_e(2)/Z_e(1), Z_e(3)/Z_e(1) \to 0$ for $L \to \infty$. Then, inverting Eqs. (10)-(12) one therefore deduces

$$\begin{aligned} &-\ln(Z_k(1)) \to -\ln(1-2Z_e(1)) \sim Z_e(1)\ ,\quad T < T_c \\ &\Rightarrow\quad 1/(\xi_-^{(0)}T_c) = b_-^{(1)} = 3.87 \pm 0.5 \end{aligned} \tag{22}$$

for the string tension, which is again consistent with the value $\simeq 3.66$ implied by the universal amplitude ratio. Due to similar relations for $Z_k(2)$ and $Z_k(3)$, we expect $b_-^{(1)} = b_-^{(2)} = b_-^{(3)}$ in our fits (18) for 2 and 3 spatial 't Hooft loops below $T_c$, which is verified well within our $\mathcal{O}(10\%)$ accuracy on these amplitudes. Above $T_c$ on the other hand, we expect for the dual string tension amplitudes

$$\tilde{\sigma}_0^{(1)} : \tilde{\sigma}_0^{(2)} : \tilde{\sigma}_0^{(3)} \ \sim\ 1 : \sqrt{2} : \sqrt{3}\ , \tag{23}$$

according to the effective area of diagonal loops. In Fig. 4 such a square-root behavior is successfully enforced on the fits (18) of the spatial 't Hooft loops above $T_c$. We obtain the same ratios, with less accuracy, for the slopes of the electric-flux free energies below $T_c$, as expected for diagonal fluxes with string formation.

To summarize, we have shown that, below $T_c$, the free energy of electric fluxes diverges linearly with the length $L$ of the system. Because spatial twists share their qualitative low-temperature behavior with the temporal ones

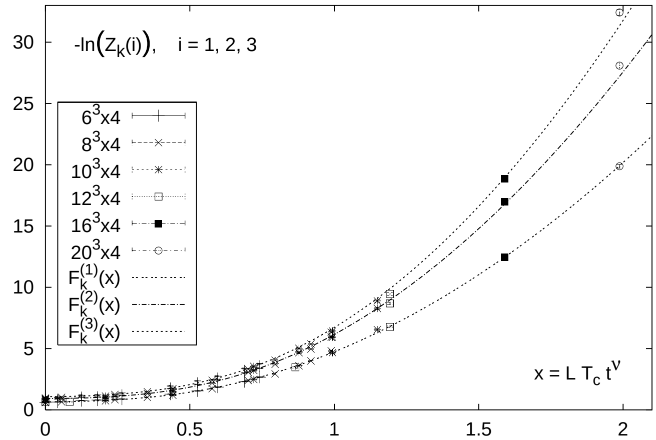


FIG. 4. Free energy of 1, 2 and 3 't Hooft loops at $T > T_c$.

considered here, the free energy of the magnetic fluxes vanishes. This therefore is the magnetic Higgs phase with electric confinement of $SU(2)$ Yang-Mills theory.

At criticality all free energies rapidly approach their finite $L \to \infty$ limits. We obtain, *e.g.*, $Z_k(1) = 0.54(1)$ for $T = T_c$, which agrees with the corresponding ratio in the 3d-Ising model [13]. This entails that the system passes through a Coulomb phase without mass gap. The fate of the magnetic charges in this phase and its self-duality as proposed in [6] remain to be further investigated.

Above $T_c$, the free energy of electric charges vanishes in the thermodynamic limit. The dual area law prevents magnetic charges from propagating in spatial directions.

The transition is well described by exponents and amplitude ratios of the 3d-Ising universality class.

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